

# Quantum entanglement and generalized information processing using entangled states with odd number of particles

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**Abstract.** We discuss and generalize multi-particle entanglement based on statistical correlations using Ursell-Mayer type of cluster coefficients. Cluster coefficients are used to distinguish different, independent entangled systems as well as those which are connected through local unitary transformations. We propose a genuinely and maximally entangled five-particle state for efficient information processing. The physical realization of entangled states and information processing protocols are analyzed using quantum gates and circuit diagrams. We show that direct as well as controlled communication can be achieved using the state proposed here, with certainty in the case of teleportation and with a high degree of optimality in the case of dense coding. For controlled dense coding the amount of information transferred from the sender to the receiver is always a maximum irrespective of the measurement basis used by the controller.

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## 1. Introduction

Quantum entanglement is a key resource for quantum information processing (QIP) protocols [1-4]. Information processing involving multi-particle states requires entangled channels which can process the information from one remote location to another with reliability. Experimental realization of multi-particle systems and the detection of all orthogonal basis states forming a complete set of entangled states remains a challenge [5-9], nevertheless, efficient theoretical construction and characterization of different multi-particle entangled channels for analyzing different information protocols is an important precursor to successful design of experiments.

Quantum teleportation involving many particles has been studied theoretically using different multi-particle entangled systems [10-22]. Many experiments have also been performed which provide partial experimental support to this concept [23-28]. Information processing protocols such as dense coding deal with sending classical information using an entangled quantum state as a shared resource [29-33]. Quantum information processing techniques through nuclear magnetic resonance have been considered in detail elsewhere [34-42].

In this article, we propose generalized multi-particle entangled systems for improving the efficiency of information processing. We do this by proposing particle correlations, as a direct measure of entanglement, using standard Ursell-Mayer terms which are firmly founded on the principles of many body statistical mechanics [43-47]. The approach presented here can be expanded and is applicable to statistical ensembles, and therefore, to electrons and other spin-1/2 systems as well as photons [48-50]. Statistical correlation coefficients are shown to be useful in distinguishing entangled systems belonging to different families. The properties of correlation coefficients are used to determine whether the states under study are related through local transformations or not. In section 3, we propose and discuss the properties of a five-particle entangled channel and generalize the quantum channel for  $(2N + 1)$  number of particles. The quantum channel proposed in that section is used for various information processing protocols successfully. This is followed by a conclusion.

## 2. Multi-particle entanglement

In this section, we first review the entanglement properties of a few maximally entangled states used in the past by others and then propose multi-particle genuinely entangled states for use in information processing. A criterion is used to define the extent of correlation between particles and several examples of entangled states of many particles are considered. The entanglement properties of bipartite states and a few multi-partite states have been studied extensively [51-60]. However, the same for the multi-particle states is not well established. Here, the extent of entanglement is assessed by the well established statistical mechanical formula for correlation coefficients [43-47]. Correlation measures for multi-particle systems defined using Ursell-Mayer type cluster coefficients

are suggested by us as a means for generalizing the defining of degree of entanglement between many particles.

### 2.1. Two and three-particle states

Correlation coefficients for two spin-1/2 particles (qubits) are defined as

$$C_{\alpha\beta}^{12} = \langle \sigma_\alpha^1 \sigma_\beta^2 \rangle - \langle \sigma_\alpha^1 \rangle \langle \sigma_\beta^2 \rangle \quad (1)$$

where  $\sigma$ 's are the Pauli spin matrices for the indicated particles,  $\{\alpha, \beta, \gamma = x, y, z\}$ . They are components of a second rank symmetric traceless tensor. The averages are calculated for the four Bell states of two entangled spin-1/2 particles, namely

$$|\psi\rangle_{12}^\pm = \frac{1}{\sqrt{2}} [ |01\rangle \pm |10\rangle ]_{12} \quad \text{and} \quad |\phi\rangle_{12}^\pm = \frac{1}{\sqrt{2}} [ |00\rangle \pm |11\rangle ]_{12} \quad (2)$$

and the non-zero correlation coefficients ( $C_{xx}^{12}$ ,  $C_{yy}^{12}$  and  $C_{zz}^{12}$ ) have the absolute value  $|1|$ . The maximum value ( $\pm 1$ ) of correlation between the particles indicates that the states are maximally entangled. The non-zero correlation coefficients ( $C_{xz}^{12}$ ,  $C_{yy}^{12}$ ,  $C_{zx}^{12}$ ) for the states

$$\begin{aligned} |\psi'\rangle_{12}^\pm &= \frac{1}{2} [ |00\rangle - |01\rangle \pm |10\rangle \pm |11\rangle ]_{12} \quad \text{and} \\ |\phi'\rangle_{12}^\pm &= \frac{1}{2} [ |00\rangle + |01\rangle \pm |10\rangle \mp |11\rangle ]_{12} \end{aligned} \quad (3)$$

which can be obtained by doing a Hadamard operation on the  $2^{nd}$  particle of Bell states in Eq. (2), show that they are also maximally entangled. The value of all correlation coefficients associated with states such as  $|\psi\rangle_{12} = \frac{1}{2} [ |00\rangle + |01\rangle + |10\rangle + |11\rangle ]_{12}$  are zero. It is evident because  $|\psi\rangle_{12} = \frac{1}{\sqrt{2}} [ |0\rangle + |1\rangle ]_1 \otimes \frac{1}{\sqrt{2}} [ |0\rangle + |1\rangle ]_2$ , a direct product state of particle 1 and particle 2. Also the existence of the maximum value for a single correlation coefficient alone does not ensure that a given system is maximally entangled, e.g. a two-particle system in a mixed state with its density operator given by  $\rho^{12} = \frac{1}{2} [ |00\rangle_{12} \langle 00|_{12} + |11\rangle_{12} \langle 11|_{12} ]$  shows  $C_{zz}^{12} = 1$ , though the two particles are not entangled. They are nevertheless correlated in the sense that measurement results for spin 1 and spin 2 are not independent of each other. However, there is no “quantum” correlations which is due to the off-diagonal components  $|00\rangle_{12} \langle 11|_{12}$  and  $|11\rangle_{12} \langle 00|_{12}$  and which is the characteristic of the entangled particles. Thus, to ensure maximum entanglement, more than one information is needed i.e. either more than one statistical data should be available with respect to non-zero correlation-coefficients or the state in question must be pure along with at least one non-zero correlation coefficient with maximum value [61]. The fact that the four Bell-states are pure and possess more than one non-zero correlation coefficients shows that the correlations between the particles are quantum.

Correlation coefficients for the three-particle systems are represented as

$$\begin{aligned} C_{\alpha\beta\gamma}^{123} &= \langle \sigma_\alpha^1 \sigma_\beta^2 \sigma_\gamma^3 \rangle - \langle \sigma_\alpha^1 \rangle \langle \sigma_\beta^2 \sigma_\gamma^3 \rangle - \langle \sigma_\beta^2 \rangle \langle \sigma_\alpha^1 \sigma_\gamma^3 \rangle - \langle \sigma_\gamma^3 \rangle \langle \sigma_\alpha^1 \sigma_\beta^2 \rangle \\ &\quad + 2 \langle \sigma_\alpha^1 \rangle \langle \sigma_\beta^2 \rangle \langle \sigma_\gamma^3 \rangle. \end{aligned} \quad (4)$$

They are components of a third rank tensor. The non-zero correlation coefficients  $C_{\alpha\beta\gamma}^{123}$  for the three-particle Greenberger-Horne-Zeilinger (GHZ) states [5], given by

$$\begin{aligned} |\psi\rangle_{123}^{(1),(2)} &= \frac{1}{\sqrt{2}} [ |000\rangle \pm |111\rangle ]_{123}, & |\psi\rangle_{123}^{(3),(4)} &= \frac{1}{\sqrt{2}} [ |001\rangle \pm |110\rangle ]_{123}, \\ |\psi\rangle_{123}^{(5),(6)} &= \frac{1}{\sqrt{2}} [ |010\rangle \pm |101\rangle ]_{123} \text{ and } |\psi\rangle_{234}^{(7),(8)} &= \frac{1}{\sqrt{2}} [ |011\rangle \pm |100\rangle ]_{123} \end{aligned} \quad (5)$$

are either +1 or -1 for the coefficients  $(C_{xxx}^{123}, C_{yyx}^{123}, C_{xyx}^{123}, C_{xyy}^{123})$ . The values suggest that the correlations between three particles are genuine and quantum. The three-particle GHZ states, though maximally entangled, are not robust with respect to disposal of any of the particles i.e. tracing of any of the particles results in the disappearance of quantum correlation between the rest of the particles. The other popular three-particle entangled state is W state [62], given by

$$|\psi\rangle_{123}^W = \frac{1}{\sqrt{3}} [ |001\rangle + |010\rangle + |100\rangle ]_{123} \quad (6)$$

have the value  $\sim 4/9$  for the non-zero correlation coefficients  $(C_{xxz}^{123}, C_{xzx}^{123}, C_{yyz}^{123}, C_{yzy}^{123}, C_{zxx}^{123}, C_{zyy}^{123}, C_{zzz}^{123})$  which suggests that the correlation between three particles is less than the maximum. The state is robust with respect to tracing of any of the particles. A similar calculation of correlation coefficients for a set of states such as

$$\begin{aligned} |\zeta\rangle_{123}^{(1),(2)} &= \frac{|\phi\rangle_{13}^+ \otimes |0\rangle_2 \pm |\phi\rangle_{13}^- \otimes |1\rangle_2}{\sqrt{2}}, \\ |\zeta\rangle_{123}^{(3),(4)} &= \frac{|\phi\rangle_{13}^+ \otimes |1\rangle_2 \pm |\phi\rangle_{13}^- \otimes |0\rangle_2}{\sqrt{2}}, \\ |\zeta\rangle_{123}^{(5),(6)} &= \frac{|\psi\rangle_{13}^+ \otimes |0\rangle_2 \pm |\psi\rangle_{13}^- \otimes |1\rangle_2}{\sqrt{2}} \text{ and } \\ |\zeta\rangle_{123}^{(7),(8)} &= \frac{|\psi\rangle_{13}^+ \otimes |1\rangle_2 \pm |\psi\rangle_{13}^- \otimes |0\rangle_2}{\sqrt{2}} \end{aligned} \quad (7)$$

shows that these states are maximally entangled as well  $(C_{xxz}^{123}, C_{yyx}^{123}, C_{yzy}^{123}, C_{xyy}^{123} \text{ are non zero})$ . In angular momentum algebraic parlance states represented in Eq. (7) and GHZ states refer to different coupling schemes and can be locally transformed into each other. The entanglement properties of these states are similar to the GHZ states if we consider the extent of correlation between three particles. Thus, if the value of correlation coefficients associated with a particular system is maximum then it indicates that the state in question possesses genuine multi-particle quantum correlations and is maximally entangled. However, if the value is not maximum but more than one non-zero correlation coefficients exists the state is non-maximally entangled. For a direct product state all the correlation coefficients are zero suggesting no genuine multi-particle correlation between the particles.

The criteria to measure the degree of entanglement using statistical correlations is compared with the existing criteria's such as concurrence [52,53] (for two-particle

systems) and with 3-tangle for three-particle maximally entangled GHZ states and average value of square of the concurrence for less than maximally entangled W state [54, 62]. Concurrence for a two-particle system is defined as

$$C(|\psi\rangle) = \langle \psi | \tilde{\psi} \rangle = \langle \psi | \sigma_y | \psi^* \rangle = C_{yy}^{12}. \quad (8)$$

where  $|\tilde{\psi}\rangle = \sigma_y |\psi^*\rangle$  and  $|\psi^*\rangle$  is complex conjugate of  $|\psi\rangle$ . Above expression shows that the value of concurrence is equal to one of the coefficient  $C_{yy}^{12}$  of second rank symmetric traceless tensor representing the correlation between the particles. Table 1 summarizes the comparison between the value of concurrence and correlation coefficients obtained for Bell states.

Table 1

state	concurrence	$C_{xx}^{12}$	$C_{yy}^{12}$	$C_{zz}^{12}$
$ \psi\rangle_{12}^-$	-1	-1	-1	-1
$ \psi\rangle_{12}^+$	1	1	1	-1
$ \phi\rangle_{12}^-$	1	-1	1	1
$ \phi\rangle_{12}^+$	-1	1	-1	1

The average value of the square of the concurrence for less than maximally entangled generalized W states  $|W\rangle_N$  is given by  $\frac{4}{N^2}$ . For maximally entangled three-particle systems (ABC) such as GHZ state(s), 3-tangle is defined as

$$\tau = C_{A(BC)}^2 - C_{AB}^2 - C_{AC}^2 = 2(\lambda_1^{AB} \lambda_2^{AB} + \lambda_1^{AC} \lambda_2^{AC}) \quad (9)$$

where  $\lambda_1^{AB}$ ,  $\lambda_2^{AB}$  and  $\lambda_1^{AC}$ ,  $\lambda_2^{AC}$  are the square roots of eigen values of  $\rho^{AB} \tilde{\rho}^{AB}$  and  $\rho^{AC} \tilde{\rho}^{AC}$ , respectively such that  $\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$ . These two values are calculated and compared with that of correlation coefficients obtained using criterion used by us. The results are summarized in Table 2 and Table 3, respectively.

Table 2

state	average value of square of the concurrence	value of correlation coefficients
$ \psi\rangle_{123}^W$	$\sim 0.45$	$\sim 0.45$
$ \psi\rangle_{1234}^W$	0.25	0.25
$ \psi\rangle_{12345}^W$	0.16	$\sim 0.16$

Table 3

state	3-tangle	$C_{xxx}^{123}$	$C_{yyx}^{123}$	$C_{yxy}^{123}$	$C_{xyy}^{123}$	$C_{xzx}^{123}$	$C_{yzy}^{123}$
$\frac{1}{\sqrt{2}} [ 000\rangle \pm  111\rangle]_{123}$	1	$\pm 1$	$\mp 1$	$\mp 1$	$\mp 1$	-	-
$\frac{1}{\sqrt{2}} [ 001\rangle \pm  110\rangle]_{123}$	1	$\pm 1$	$\mp 1$	$\pm 1$	$\pm 1$	-	-
$\frac{1}{\sqrt{2}} [ 010\rangle \pm  101\rangle]_{123}$	1	$\pm 1$	$\pm 1$	$\mp 1$	$\pm 1$	-	-
$\frac{1}{\sqrt{2}} [ 011\rangle \pm  100\rangle]_{123}$	1	$\pm 1$	$\pm 1$	$\pm 1$	$\mp 1$	-	-
$ \zeta\rangle_{123}^{(1)}$	1	-	-1	-	-1	-1	+1
$ \zeta\rangle_{123}^{(2)}$	1	-	-1	-	-1	+1	-1
$ \zeta\rangle_{123}^{(3)}$	1	-	+1	-	+1	+1	-1
$ \zeta\rangle_{123}^{(4)}$	1	-	+1	-	+1	-1	+1
$ \zeta\rangle_{123}^{(5)}$	1	-	+1	-	-1	+1	+1
$ \chi\rangle_{123}^{(6)}$	1	-	+1	-	-1	-1	-1
$ \zeta\rangle_{123}^{(7)}$	1	-	-1	-	+1	-1	-1
$ \zeta\rangle_{123}^{(8)}$	1	-	-1	-	+1	+1	+1

Table 3 and Table 2 show that the value of non-zero correlation coefficients for three-particle GHZ state(s) and three-particle  $|\zeta\rangle_{123}^{(i)}$  are in excellent argument with the value of 3-tangle whereas average value of square of the concurrence for  $|W\rangle_N$  is also a match with the value of non-zero correlation coefficients obtained. This suggests that the criterion using statistical correlation coefficients to measure the degree of entanglement include all possible type of entanglement in multi-particle systems and is a noble idea to study and analyze the properties of multi-particle systems. This can thus be generalized for arbitrary number of particles.

## 2.2. Four particle systems

The expression for four-particle correlation coefficients is given by

$$\begin{aligned}
C_{\alpha\beta\gamma\delta}^{1234} = & \langle \sigma_\alpha^1 \sigma_\beta^2 \sigma_\gamma^3 \sigma_\delta^4 \rangle - \langle \sigma_\alpha^1 \rangle [C_{\beta\gamma\delta}^{234}] - \langle \sigma_\beta^2 \rangle [C_{\alpha\gamma\delta}^{134}] - \langle \sigma_\gamma^3 \rangle [C_{\alpha\beta\delta}^{124}] \\
& - \langle \sigma_\delta^4 \rangle [C_{\alpha\beta\gamma}^{123}] - \langle \sigma_\alpha^1 \sigma_\beta^2 \rangle \langle \sigma_\gamma^3 \sigma_\delta^4 \rangle - \langle \sigma_\alpha^1 \sigma_\gamma^3 \rangle \langle \sigma_\beta^2 \sigma_\delta^4 \rangle \\
& - \langle \sigma_\alpha^1 \sigma_\delta^4 \rangle \langle \sigma_\beta^2 \sigma_\gamma^3 \rangle + 2 \langle \sigma_\alpha^1 \rangle \langle \sigma_\beta^2 \rangle \langle \sigma_\gamma^3 \rangle \langle \sigma_\delta^4 \rangle
\end{aligned} \tag{10}$$

The non-zero correlation coefficients calculated for the four-particle GHZ states, namely

$$|\psi\rangle_{1234}^{GHZ} = \frac{1}{\sqrt{2}} [|n_1 n_2 n_3 n_4\rangle \pm |n'_1 n'_2 n'_3 n'_4\rangle] \tag{11}$$

where if  $n_i = 0$  then  $n'_i = 1$  and *vice versa* are  $C_{xxx}^{1234}$ ,  $C_{xyy}^{1234}$ ,  $C_{xyx}^{1234}$ ,  $C_{yyx}^{1234}$ ,  $C_{yyy}^{1234}$ ,  $C_{zzz}^{1234}$  and indicate that four-particle GHZ states possess maximum correlations. Similarly, the non-zero correlation coefficients calculated for the four-particle W state,  $|\psi\rangle_{1234}^W = \frac{1}{2} [|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle]_{1234}$ , are  $C_{xxx}^{1234}$ ,  $C_{xxz}^{1234}$ ,  $C_{zzx}^{1234}$ ,  $C_{yyy}^{1234}$ ,  $C_{yyz}^{1234}$ ,  $C_{zyy}^{1234}$ ,  $C_{yzy}^{1234}$ ,  $C_{zyz}^{1234}$ ,  $C_{zzx}^{1234}$ ,  $C_{zxx}^{1234}$ ,  $C_{xzz}^{1234}$ ,  $C_{zzy}^{1234}$ ,  $C_{zyz}^{1234}$ ,  $C_{zzz}^{1234}$  and show the value as ( $\sim 1/4$ ) indicating less than maximum correlations between particles. Rigolin [17] proposed a generalized Bell basis as a set of four-particle states to be used for information processing, however, all the 16 four-particle correlation coefficients associated with the generalized Bell basis are zero suggesting that there is no genuine correlation between the four-particles. Yeo and Chua [20] proposed a four-particle entangled system  $|\chi\rangle_{1234}^{00}$ ; the maximum value of non-zero correlation coefficients  $C_{xyy}^{1234}$ ,  $C_{xxx}^{1234}$ ,  $C_{zyy}^{1234}$  and  $C_{zzz}^{1234}$  indicates that the state is maximally and genuinely entangled. We consider here three sets of four-particle maximally entangled states, in addition to GHZ states, given by  $|\phi\rangle_{1234}^{(1)-(16)}$ ,  $|\chi\rangle_{1234}^{(1)-(16)}$  and  $|\phi'\rangle_{1234}^{(1)-(16)}$  where

$$|\phi\rangle_{1234}^{(1)-(16)} = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} |0\rangle_1 \\ |1\rangle_1 \end{pmatrix} \otimes \begin{pmatrix} |\phi^+\rangle_{24} \\ |\psi^+\rangle_{24} \end{pmatrix} \otimes \begin{pmatrix} |0\rangle_3 \\ |1\rangle_3 \end{pmatrix} \right. \\ \left. \pm \begin{pmatrix} |1\rangle_1 \\ |0\rangle_1 \end{pmatrix} \otimes \begin{pmatrix} |\phi^-\rangle_{24} \\ |\psi^-\rangle_{24} \end{pmatrix} \otimes \begin{pmatrix} |1\rangle_3 \\ |0\rangle_3 \end{pmatrix} \right], \quad (12)$$

$$|\chi\rangle_{1234}^{(1)-(16)} = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} |0\rangle_1 \\ |1\rangle_1 \end{pmatrix} \otimes \begin{pmatrix} |\phi^+\rangle_{24} \\ |\phi^-\rangle_{24} \end{pmatrix} \otimes \begin{pmatrix} |0\rangle_3 \\ |1\rangle_3 \end{pmatrix} \right. \\ \left. \pm \begin{pmatrix} |1\rangle_1 \\ |0\rangle_1 \end{pmatrix} \otimes \begin{pmatrix} |\psi^-\rangle_{24} \\ |\psi^+\rangle_{24} \end{pmatrix} \otimes \begin{pmatrix} |1\rangle_3 \\ |0\rangle_3 \end{pmatrix} \right], \quad (13)$$

and

$$|\phi'\rangle_{1234}^{(1)-(16)} = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} |0\rangle_1 \\ |1\rangle_1 \end{pmatrix} \otimes \begin{pmatrix} |\phi^+\rangle_{24} \\ |\phi^-\rangle_{24} \end{pmatrix} \otimes \begin{pmatrix} |0\rangle_3 \\ |1\rangle_3 \end{pmatrix} \right. \\ \left. \pm \begin{pmatrix} |1\rangle_1 \\ |0\rangle_1 \end{pmatrix} \otimes \begin{pmatrix} |\psi^+\rangle_{24} \\ |\psi^-\rangle_{24} \end{pmatrix} \otimes \begin{pmatrix} |1\rangle_3 \\ |0\rangle_3 \end{pmatrix} \right]. \quad (14)$$

The non-zero correlation coefficients calculated for the above three sets are  $(C_{xyy}^{1234}, C_{xyx}^{1234}, C_{yyx}^{1234} \text{ and } C_{yyy}^{1234})$ ,  $(C_{xxx}^{1234}, C_{xxz}^{1234}, C_{yxy}^{1234}, C_{yzy}^{1234})$  and  $(C_{xzy}^{1234}, C_{xyy}^{1234}, C_{yzy}^{1234}, C_{yyx}^{1234})$ , respectively and indicate maximum entanglement. The set of states represented by Eq. (12) and Eq. (14) are cluster type of states [63] and can be transformed into each other through local transformations whereas Eq. (13) represents  $|\chi\rangle$  type of states [20].

### 2.3. Five-particle systems

The expression for the five-particle correlation coefficient is given in the Appendix A. The generalized five-particle GHZ states are represented as

$$|\psi\rangle_{12345}^{GHZ} = \frac{1}{\sqrt{2}} [|n_1 n_2 n_3 n_4 n_5\rangle \pm |n'_1 n'_2 n'_3 n'_4 n'_5\rangle] \quad (15)$$

and are maximally correlated as shown by non-zero correlation coefficients  $C_{xxxx}^{12345}, C_{xxxxy}^{12345}, C_{xxxyx}^{12345}, C_{xyxyx}^{12345}, C_{xyxyy}^{12345}, C_{xyyx}^{12345}, C_{xyyy}^{12345}, C_{yxxx}^{12345}, C_{yxyx}^{12345}, C_{yxyy}^{12345}, C_{yyxx}^{12345}, C_{yyxy}^{12345}, C_{yyyx}^{12345}$  and  $C_{yyyy}^{12345}$ . Unlike the GHZ state, the generalized five-particle W state,  $|\psi\rangle_{12345}^W = \frac{1}{\sqrt{5}} [|00001\rangle + |00010\rangle + |00100\rangle + |01000\rangle + |10000\rangle]_{12345}$ , is not maximally correlated as shown by non-zero correlation coefficients  $C_{xxxx}^{12345}, C_{xxxzx}^{12345}, C_{xxxzx}^{12345}, C_{xxxzx}^{12345}, C_{yyzzz}^{12345}, C_{yzyzz}^{12345}, C_{yzzzy}^{12345}, C_{yzzzy}^{12345}, C_{zxzzz}^{12345}, C_{zxzzz}^{12345}, C_{zxzzz}^{12345}, C_{zyyzz}^{12345}, C_{zyyzz}^{12345}, C_{zyyzz}^{12345}, C_{zzxxz}^{12345}, C_{zzxxz}^{12345}, C_{zzyyz}^{12345}, C_{zzyyz}^{12345}, C_{zzzzx}^{12345}, C_{zzzzx}^{12345}$  and  $C_{zzzz}^{12345}$ . Other five-particle entangled systems to be considered are two sets of basis states given as  $|\Psi\rangle_{12345}^{(1)-(32)}$  and  $|\Phi\rangle_{12345}^{(1)-(32)}$ , where

$$|\Psi\rangle_{12345}^{(1)-(32)} = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} |0\rangle_1 \\ |1\rangle_1 \end{pmatrix} \otimes \begin{pmatrix} |\psi^{(1)}\rangle_{234} \\ |\psi^{(2)}\rangle_{234} \\ |\psi^{(3)}\rangle_{234} \\ |\psi^{(4)}\rangle_{234} \end{pmatrix} \otimes \begin{pmatrix} |0\rangle_5 \\ |1\rangle_5 \end{pmatrix} \right. \\ \left. \pm \begin{pmatrix} |1\rangle_1 \\ |0\rangle_1 \end{pmatrix} \otimes \begin{pmatrix} |\psi^{(6)}\rangle_{234} \\ |\psi^{(5)}\rangle_{234} \\ |\psi^{(8)}\rangle_{234} \\ |\psi^{(7)}\rangle_{234} \end{pmatrix} \otimes \begin{pmatrix} |1\rangle_5 \\ |0\rangle_5 \end{pmatrix} \right] \quad (16)$$

and

$$|\Phi\rangle_{12345}^{(1)-(32)} = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} |0\rangle_1 \\ |1\rangle_1 \end{pmatrix} \otimes \begin{pmatrix} |\psi^{(1)}\rangle_{234} \\ |\psi^{(2)}\rangle_{234} \\ |\psi^{(3)}\rangle_{234} \\ |\psi^{(4)}\rangle_{234} \end{pmatrix} \otimes \begin{pmatrix} |0\rangle_5 \\ |1\rangle_5 \end{pmatrix} \right. \\ \left. \pm \begin{pmatrix} |1\rangle_1 \\ |0\rangle_1 \end{pmatrix} \otimes \begin{pmatrix} |\psi^{(5)}\rangle_{234} \\ |\psi^{(6)}\rangle_{234} \\ |\psi^{(7)}\rangle_{234} \\ |\psi^{(8)}\rangle_{234} \end{pmatrix} \otimes \begin{pmatrix} |1\rangle_5 \\ |0\rangle_5 \end{pmatrix} \right]. \quad (17)$$

$|\psi\rangle_{234}^{(1)-(8)}$  are three-particle GHZ states and are given by Eq. (5). The non-zero correlation coefficients for the two sets are  $C_{xxxx}^{12345}, C_{xyzyx}^{12345}, C_{yxzxy}^{12345}, C_{yyzyy}^{12345}$  and  $C_{xxzyy}^{12345}, C_{xyzyx}^{12345}, C_{yxzxy}^{12345}, C_{yyzyy}^{12345}$ , respectively and show maximum value. The extent of correlation between five-particles remains the same even after interchanging the particle indices.

The general expression for the  $N$ -particle correlation coefficient can be obtained by solving the equations for cluster functions derived formally from the  $N$ -th quantum virial coefficient. The following summarizes the relation between correlation coefficients and the degree of entanglement.

- (i) Existence of maximum values for more than one correlation coefficient for a system under study, indicates that the state of the system possesses genuine and maximum entanglement.
- (ii) For non-maximally entangled states the value of correlation coefficients lies between



0 and 1.

- (iii) Null results for all correlation coefficients of state suggest that it is a direct product of fewer particle states and there exists no genuine multi-particle entanglement.
- (iv) The value of non-zero correlation coefficients remains the same for states connected to each other by local unitary transformations.
- (v) The extent of correlation remains invariant to changing the particle indices.

#### 2.4. Importance and properties of cluster coefficients

The criterion to use cluster coefficients as a measure of entanglement of the state under study allows one to characterize the extent of correlation of multi-particle states on the same scale irrespective of number of particles involved. A consistent description emerges for systems irrespective of the number of particles which are entangled. In this subsection, we discuss some of the properties of correlation coefficients in addition to those described in previous section.

- (i) The relation between correlation coefficients of states which differ from each other only through permutation of particle indices can be seen immediately as follows:
  - (1) The state  $|\phi\rangle_{1234}^{(1)} = \frac{1}{2} [|0000\rangle + |0101\rangle + |1010\rangle - |1111\rangle]_{1234}$  [Eq. (12)] is obtained from  $|\phi''\rangle_{1234}^{(1)} = \frac{1}{2} [|0000\rangle + |1001\rangle + |0110\rangle - |1111\rangle]_{1234}$  by permuting particles 1 and 2. Hence, the non-zero correlation coefficients associated with  $|\phi\rangle_{1234}^{(1)}$  and  $|\phi''\rangle_{1234}^{(1)}$  are  $(C_{xxyy}^{1234}, C_{xyyx}^{1234}, C_{yyxy}^{1234}, C_{yyxx}^{1234})$  and  $(C_{xxyy}^{1234}, C_{xyyx}^{1234}, C_{yyxy}^{1234}, C_{yyxx}^{1234})$ , respectively.
  - (2) Conversely, by examining two sets of equal number of correlation coefficients, we can also relate the states. For example, the two sets  $(C_{xxxx}^{12345}, C_{xyzy}^{12345}, C_{yxzx}^{12345}, C_{yyzy}^{12345})$  and  $(C_{xxxx}^{12345}, C_{yyxz}^{12345}, C_{xyyz}^{12345}, C_{yyyz}^{12345})$  are related to each other through particle permutations  $(1 \leftrightarrow 2)$  and  $(3 \leftrightarrow 5)$ . The first set is the only non-null set of coefficients for the state given by Eq. (16). Hence another can be obtained by particle permutations. Thus, a family of states can be quickly enumerated.
- (ii) If the number of non-zero correlation coefficients corresponding to two entangled sets are not equal, then they belong to two different family of states.
  - (1) Three-particle GHZ state  $(C_{xxx}^{123}, C_{yyx}^{123}, C_{yxy}^{123}, C_{xyy}^{123})$  and three-particle W state  $(C_{xxz}^{123}, C_{xzx}^{123}, C_{yyz}^{123}, C_{yzy}^{123}, C_{zxx}^{123}, C_{zyy}^{123}, C_{zzz}^{123})$  show four and seven non-zero correlation coefficients, respectively. They belong to two different families of states.
  - (2) Four-particle maximally entangled GHZ states  $(C_{xxxx}^{1234}, C_{xxyy}^{1234}, C_{xyxy}^{1234}, C_{xyyx}^{1234}, C_{yyxx}^{1234}, C_{yyxy}^{1234}, C_{yyxx}^{1234}, C_{yyyy}^{1234}, C_{zzzz}^{1234})$  represented by Eq. (11) and four-particle maximally entangled set represented by Eq. (12)  $(C_{xxyy}^{1234}, C_{yyxy}^{1234}, C_{xyyx}^{1234}, C_{yyxx}^{1234})$  show nine and four non-zero correlation coefficients, respectively which indicates that these two sets belong to different families of states.
  - (3) The set of five-particle states represented by Eq. (17)  $(C_{xxzy}^{12345}, C_{xyzy}^{12345}, C_{yxzy}^{12345}, C_{yyzx}^{12345})$  and five-particle Brown state [16]  $(C_{xyyx}^{12345}, C_{xxzx}^{12345}, C_{yyzz}^{12345}, C_{yzxy}^{12345}, C_{zxzy}^{12345}, C_{zyxz}^{12345})$  possess four and six non-zero correlation coefficients, respectively and hence belong to two different families of entangled systems.

- (iii) Even if the number of correlation coefficients associated with different entangled states are equal the states need not have to belong to the same family. For example, the three states  $|\phi\rangle_{1234}^{(1)} = \frac{1}{2} [|0000\rangle + |0101\rangle + |1010\rangle - |1111\rangle]_{1234}$ ,  $|\chi\rangle_{1234}^{(1)} = \frac{1}{2} [|0000\rangle + |0101\rangle + |1011\rangle - |1110\rangle]_{1234}$  and  $|\phi'\rangle_{1234}^{(1)} = \frac{1}{2} [|0000\rangle + |0101\rangle + |1011\rangle + |1110\rangle]_{1234}$  belong to maximally entangled four-particle sets represented by Eq. (12), Eq. (13) and Eq. (14), respectively and possess four non-zero correlation coefficients. Although  $|\phi\rangle_{1234}^{(1)}$  and  $|\phi'\rangle_{1234}^{(1)}$  belong to same family,  $|\chi\rangle_{1234}^{(1)}$  belongs to different family of states.
- (iv) The extent of correlation between particles remains invariant under standard local unitary transformations. The set of states ( $|\zeta\rangle_{123}^{(i)}$ ) represented by Eq. (7) can be obtained by applying Hadamard operation to the second particle of three-particle GHZ states (Eq. (5)) and possesses the same degree of correlation as that of GHZ states.
- (v) Doing a Hadamard operation would not affect the y-component of the correlation coefficient but would convert x-component to z-component and *vice versa*. Thus, the non-zero correlation coefficients for the set of states represented by Eq. (2) and Eq. (3) (which differ from each other by a Hadamard transformation on particle 2) are  $(C_{xx}^{12}, C_{yy}^{12}, C_{zz}^{12})$  and  $(C_{xz}^{12}, C_{yy}^{12}, C_{zx}^{12})$ , respectively. This is a trivial example. However, as the number of particles in an entangled set increases, the number of ways of doing transformations also increases and hence this scheme is useful for nontrivial, multiple local transformations as shown below.
- (1) The correlation coefficients,  $(C_{xyyx}^{1234}, C_{xxzx}^{1234}, C_{zyyz}^{1234}, C_{zzzz}^{1234})$  and  $(C_{yxyz}^{1234}, C_{xxzx}^{1234}, C_{yzyx}^{1234}, C_{xxxx}^{1234})$ , correspond to the entangled states  $|\chi\rangle_{1234}^{00} = \frac{1}{2\sqrt{2}} [|0000\rangle - |0011\rangle - |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle + |1111\rangle]_{1234}$  and  $|\chi\rangle_{1234}^{(1)} = \frac{1}{2} [|0000\rangle + |0101\rangle + |1011\rangle - |1110\rangle]_{1234}$ , respectively. The set of coefficients can be transformed into each other by doing Hadamard transformations on  $2^{nd}$ ,  $3^{rd}$  and  $4^{th}$  particles and permuting particles 1 and 2.
- (2) The five-particle maximally entangled states, namely  $|\varsigma\rangle_{12345}^{(1)} = \frac{1}{2} [|00000\rangle + |01110\rangle + |10001\rangle - |11111\rangle]_{12345}$  and  $|\varsigma'\rangle_{12345}^{(1)} = \frac{1}{2\sqrt{2}} [|00000\rangle + |00110\rangle + |01010\rangle + |01100\rangle + |10011\rangle + |10101\rangle + |11001\rangle + |11111\rangle]_{12345}$  can be converted into one another by doing local transformations as revealed by their correlation coefficients, namely  $(C_{xxxyy}^{12345}, C_{xyxyy}^{12345}, C_{xyxxy}^{12345}, C_{xyyyy}^{12345}, C_{xzzzz}^{12345}, C_{yxxxy}^{12345}, C_{yxyxx}^{12345}, C_{yyxxx}^{12345}, C_{yyyxy}^{12345}, C_{yzzzy}^{12345})$  and  $(C_{xxzyy}^{12345}, C_{xzyzy}^{12345}, C_{xyzzz}^{12345}, C_{xyyyy}^{12345}, C_{xxxxx}^{12345}, C_{yzzzy}^{12345}, C_{yzzyx}^{12345}, C_{yyzzx}^{12345}, C_{yyyxy}^{12345}, C_{yxxxxy}^{12345})$ , respectively. Thus by doing three Hadamard operations on particle two, three and four  $|\varsigma\rangle_{12345}^{(1)}$  can be locally transformed to  $|\varsigma'\rangle_{12345}^{(1)}$ .

### 3. Generalized information processing

In this section we propose a maximally and genuinely entangled five-particle state and describe different information processing protocols using the state. In the past, multi-particle entangled channels involving odd number of particles have been proposed *with*

the use of a controller to assist the sender for successful and optimal information transfer [10, 13, 14]. We show that one can *eliminate the intermediate observer* controlling the process such that information processing is successful in all the measurement outcomes performed by the sender. The formation of the state proposed here ensures efficient information transfer between two or more users in the communication protocol.

### 3.1. Direct teleportation

The five-particle maximally entangled set proposed here is given by

$$|\varphi\rangle_{12345}^{(1)-(32)} = \frac{1}{\sqrt{2}} \left[ \left( \begin{array}{c} |\chi^{(1)}\rangle_{1234} \\ |\chi^{(2)}\rangle_{1234} \\ |\chi^{(3)}\rangle_{1234} \\ |\chi^{(4)}\rangle_{1234} \\ |\chi^{(5)}\rangle_{1234} \\ |\chi^{(6)}\rangle_{1234} \\ |\chi^{(7)}\rangle_{1234} \\ |\chi^{(8)}\rangle_{1234} \end{array} \right) \otimes \left( \begin{array}{c} |0\rangle_5 \\ |1\rangle_5 \end{array} \right) \pm \left( \begin{array}{c} |\chi^{(9)}\rangle_{1234} \\ |\chi^{(10)}\rangle_{1234} \\ |\chi^{(11)}\rangle_{1234} \\ |\chi^{(12)}\rangle_{1234} \\ |\chi^{(13)}\rangle_{1234} \\ |\chi^{(14)}\rangle_{1234} \\ |\chi^{(15)}\rangle_{1234} \\ |\chi^{(16)}\rangle_{1234} \end{array} \right) \otimes \left( \begin{array}{c} |1\rangle_5 \\ |0\rangle_5 \end{array} \right) \right] \quad (18)$$

where  $|\chi\rangle_{1234}^{(1)-(16)}$  are given by

$$\begin{aligned} |\chi\rangle_{1234}^{(1),(2)} &= \frac{|0\rangle_1 \otimes |\phi\rangle_{24}^+ \otimes |0\rangle_3 \pm |1\rangle_1 \otimes |\psi\rangle_{24}^- \otimes |1\rangle_3}{\sqrt{2}}, \\ |\chi\rangle_{1234}^{(3),(4)} &= \frac{|0\rangle_1 \otimes |\phi\rangle_{24}^- \otimes |0\rangle_3 \pm |1\rangle_1 \otimes |\psi\rangle_{24}^+ \otimes |1\rangle_3}{\sqrt{2}}, \\ |\chi\rangle_{1234}^{(5),(6)} &= \frac{|1\rangle_1 \otimes |\phi\rangle_{24}^+ \otimes |0\rangle_3 \pm |0\rangle_1 \otimes |\psi\rangle_{24}^- \otimes |1\rangle_3}{\sqrt{2}}, \\ |\chi\rangle_{1234}^{(7),(8)} &= \frac{|1\rangle_1 \otimes |\phi\rangle_{24}^- \otimes |0\rangle_3 \pm |0\rangle_1 \otimes |\psi\rangle_{24}^+ \otimes |1\rangle_3}{\sqrt{2}}, \\ |\chi\rangle_{1234}^{(9),(10)} &= \frac{|0\rangle_1 \otimes |\phi\rangle_{24}^+ \otimes |1\rangle_3 \mp |1\rangle_1 \otimes |\psi\rangle_{24}^- \otimes |0\rangle_3}{\sqrt{2}}, \\ |\chi\rangle_{1234}^{(11),(12)} &= \frac{|0\rangle_1 \otimes |\phi\rangle_{24}^- \otimes |1\rangle_3 \mp |1\rangle_1 \otimes |\psi\rangle_{24}^+ \otimes |0\rangle_3}{\sqrt{2}}, \\ |\chi\rangle_{1234}^{(13),(14)} &= \frac{|1\rangle_1 \otimes |\phi\rangle_{24}^+ \otimes |1\rangle_3 \mp |0\rangle_1 \otimes |\psi\rangle_{24}^- \otimes |0\rangle_3}{\sqrt{2}} \text{ and} \\ |\chi\rangle_{1234}^{(15),(16)} &= \frac{|1\rangle_1 \otimes |\phi\rangle_{24}^- \otimes |1\rangle_3 \mp |0\rangle_1 \otimes |\psi\rangle_{24}^+ \otimes |0\rangle_3}{\sqrt{2}}. \end{aligned} \quad (19)$$

The set represented above is same as four-particle entangled set given in Eq. (13), however, the order in which the states are represented is different. The set proposed

here shows values  $\pm 1$  for the non-zero correlation coefficients ( $C_{xxxx}^{12345}$ ,  $C_{xxzz}^{12345}$ ,  $C_{zzxx}^{12345}$ ,  $C_{zzzz}^{12345}$ ). Depending on the discussions of previous section and due to the absence of 'y' in the  $C$ 's the five-particle entangled set proposed above belongs to a different family of states with respect to those represented by Eq. (16) and Eq. (17).

In order to communicate an arbitrary two-particle information to Bob i.e.  $|\phi\rangle_{12} = [a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle]_{12}$ , Alice must share any one of the five-particle entangled state  $|\varphi\rangle_{34567}^{(i)}$  given by Eq. (18) with Bob such that particles 3, 4 and 5 are with Alice and particles 6 and 7 are with Bob. Thus, using  $|\varphi\rangle_{34567}^{(10)}$  as the quantum channel shared between Alice and Bob where  $|\varphi\rangle_{34567}^{(10)} = \frac{1}{2\sqrt{2}} [|00000\rangle - |00101\rangle + |11100\rangle + |11001\rangle + |01111\rangle - |01010\rangle + |10011\rangle + |10110\rangle]_{34567}$ , Alice can communicate her unknown message with Bob by interacting her particles 1 and 2 with her share of entangled particles 3, 4 and 5 so that

$$|\psi\rangle_{1234567} = |\phi\rangle_{12} \otimes |\varphi\rangle_{34567}^{(10)}. \quad (20)$$

Eq. (20) can be re-expressed in form of Alice's projection basis given by Eq. (17) as

$$|\psi\rangle_{1234567} = \frac{1}{4\sqrt{2}} \sum_{i,j} |\Phi\rangle_{12345}^{(i)} \otimes |\phi\rangle_{67}^{(j)} \quad (21)$$

where  $i = 1, 32$  and  $j = 1, 4$ . For Alice's measurement outcomes  $|\Phi\rangle_{12345}^{(1)}$  and  $|\Phi\rangle_{12345}^{(30)}$ , Bob's particles are instantaneously projected on to the state Alice wanted to communicate, with total probability of  $1/16$ , however, for all other measurement outcomes of Alice, Bob require *only* single qubit transformations to recover the message successfully. The preparation of above set of states and teleportation of an arbitrary two-particle state are represented in Fig. (1) and Fig. (2), respectively.

### 3.2. Controlled teleportation

For controlled teleportation, the quantum state  $|\varphi\rangle_{34567}^{(10)}$  is shared between Alice, Charlie and Bob such that the particles 3 and 4 are with Alice, particles 5 and 6 are with Bob and particle 7 is with Charlie. Alice projects her four particles on to the basis set given by Eq. (13) so that Eq. (20) becomes

$$|\psi\rangle_{1234567} = \frac{1}{4} \sum_{i,j} |\chi\rangle_{1234}^{(i)} \otimes |\psi\rangle_{567}^{(j)} \quad (22)$$

where  $i = 1, 16$  and  $j = 1, 8$ . For example, if Alice's measurement outcome is  $|\chi\rangle_{1234}^{(5)}$ , the combined state of Bob's and Charlie's particles is given by

$$\begin{aligned} |\psi\rangle_{567} &= \frac{1}{\sqrt{2}} [a|00\rangle_{56} + b|01\rangle_{56} + c|10\rangle_{56} + d|11\rangle_{56}] |0\rangle_7 \\ &+ \frac{1}{\sqrt{2}} [-a|10\rangle_{56} - b|11\rangle_{56} + c|00\rangle_{56} + d|01\rangle_{56}] |1\rangle_7. \end{aligned} \quad (23)$$

For Charlie's outcome of  $|0\rangle_7$ , Bob's particles are in the state identical to the one communicated by Alice, however, for his outcome  $|1\rangle_7$ , Bob needs to do a  $\sigma_z^5$  and  $\sigma_x^5$  operation on the 5<sup>th</sup>-particle to complete the process successfully. Again, for all the

outcomes of Alice and Charlie, Bob can recover the message with single qubit unitary transformations, if needed. The above processes can be generalized for the case of  $(2N + 1)$  number of particles as follows.

The generalized entangled basis set corresponding to Eq. (18) is

$$|\varphi\rangle_{12\dots 2N(2N+1)}^{(1)-(2^{2N+1})} = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} |\chi^{(1)}\rangle_{12\dots(2N-1)2N} \\ |\chi^{(2)}\rangle_{12\dots(2N-1)2N} \\ \vdots \\ |\chi^{(2^{2N-1}-1)}\rangle_{12\dots(2N-1)2N} \\ |\chi^{(2^{2N-1})}\rangle_{12\dots(2N-1)2N} \end{pmatrix} \otimes \begin{pmatrix} |0\rangle_{2N+1} \\ |1\rangle_{2N+1} \end{pmatrix} \right. \\ \left. \pm \begin{pmatrix} |\chi^{(2^{2N-1}+1)}\rangle_{12\dots(2N-1)2N} \\ |\chi^{(2^{2N-1}+2)}\rangle_{12\dots(2N-1)2N} \\ \vdots \\ |\chi^{(2^{2N}-1)}\rangle_{12\dots(2N-1)2N} \\ |\chi^{(2^{2N})}\rangle_{12\dots(2N-1)2N} \end{pmatrix} \otimes \begin{pmatrix} |1\rangle_{2N+1} \\ |0\rangle_{2N+1} \end{pmatrix} \right] \quad (24)$$

where  $|\chi\rangle_{12\dots(2N-1)2N}^{(i)}$ 's are ordered in the same way as in Eq. (19) and  $|\chi\rangle_{12\dots(2N-1)2N}^{(i)}$ 's are  $2N$ -particle generalization of Eq. (13), namely

$$|\chi\rangle_{12\dots(2N-1)2N}^{(1)-(2^{2N})} = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} |0\rangle_1 \\ |1\rangle_1 \end{pmatrix} \otimes \begin{pmatrix} |\chi^{(1)}\rangle_{23\dots 2N} \\ |\chi^{(2)}\rangle_{23\dots 2N} \\ \vdots \\ |\chi^{(2^{2N-3}-1)}\rangle_{23\dots 2N} \\ |\chi^{(2^{2N-3})}\rangle_{23\dots 2N} \end{pmatrix} \otimes \begin{pmatrix} |0\rangle_{N+1} \\ |1\rangle_{N+1} \end{pmatrix} \right. \\ \left. \pm \begin{pmatrix} |1\rangle_1 \\ |0\rangle_1 \end{pmatrix} \otimes \begin{pmatrix} |\chi^{(2^{2N-3}+1)}\rangle_{23\dots 2N} \\ |\chi^{(2^{2N-3}+2)}\rangle_{23\dots 2N} \\ \vdots \\ |\chi^{(2^{2N-2}-1)}\rangle_{23\dots 2N} \\ |\chi^{(2^{2N-2})}\rangle_{23\dots 2N} \end{pmatrix} \otimes \begin{pmatrix} |1\rangle_{N+1} \\ |0\rangle_{N+1} \end{pmatrix} \right] \quad (25)$$

This can be used for teleportation of  $N$ -particle arbitrary information from Alice to Bob. The  $(2N + 1)$ -particle channel is shared between Alice and Bob such that the first  $(N + 1)$  particles are with Alice and the rest  $N$  particles are with Bob. It is important to choose the correct projection basis such that the teleportation becomes feasible in all outcomes with only single qubit operations on Bob's end. For controlled teleportation the same basis set is shared between Alice, Charlie and Bob such that the first  $N$  particles are with Alice,  $(2N + 1)$ -th particle is with Charlie and rest  $N$  particle are with Bob. To realize successful teleportation Alice projects her particle's on the  $2N$ -particle generalized basis set given by Eq. (25). To control the process effectively Charlie measures his  $(2N + 1)$ -th particle in computational basis.

### 3.3. Direct and controlled dense coding

Dense coding is concerned with the transfer of two bits of classical message between two parties by sending only one qubit (particles) from sender to receiver provided they share a maximally entangled pair of two qubits. In this subsection we discuss dense coding protocol using maximally entangled state proposed in this article.

Alice and Bob must share a maximally entangled quantum channel given by  $|\varphi\rangle_{12345}^{(i)}$  such that the qubits 1, 2, and 3 are with Alice and qubits 4 and 5 are with Bob. Alice can locally operate her qubits to encode the desired message, using operators from the set  $(I^1, \sigma_x^1, \sigma_y^1, \sigma_z^1)$ ,  $(I^2, \sigma_x^2, \sigma_y^2, \sigma_z^2)$  and  $(I^3, \sigma_x^3, \sigma_y^3, \sigma_z^3)$  corresponding to her three qubits 1, 2 and 3, respectively. Alice sends her encoded qubits to Bob who decodes the message by an appropriate measurement on the joint five qubit state.

However, for the five-qubit quantum channel  $|\varphi\rangle_{12345}^{(i)}$ , Alice will only produce 32 orthogonal states and hence can encode a 5-bit message. The operator set that is used to prepare all the orthogonal states belonging to the entangled set is given by  $[(I^1 I^2 I^3, \sigma_z^1, \sigma_z^2, \sigma_z^3, \sigma_z^1 \sigma_z^2, \sigma_z^1 \sigma_z^3, \sigma_z^2 \sigma_z^3, \sigma_z^1 \sigma_z^2 \sigma_z^3), (\sigma_x^1, \sigma_x^1 \sigma_z^2, \sigma_x^1 \sigma_z^3, \sigma_x^1 \sigma_z^2 \sigma_z^3, \sigma_x^2, \sigma_z^1 \sigma_x^2, \sigma_x^2 \sigma_z^3, \sigma_z^1 \sigma_x^2 \sigma_z^3), (\sigma_y^1, \sigma_y^1 \sigma_z^2, \sigma_y^1 \sigma_z^3, \sigma_y^1 \sigma_z^2 \sigma_z^3, \sigma_y^2, \sigma_z^1 \sigma_y^2, \sigma_y^2 \sigma_z^3, \sigma_z^1 \sigma_y^2 \sigma_z^3), (\sigma_x^1 \sigma_z^2, \sigma_x^1 \sigma_z^2 \sigma_z^3, \sigma_x^1 \sigma_y^2, \sigma_x^1 \sigma_y^2 \sigma_z^3, \sigma_y^1 \sigma_x^2, \sigma_y^1 \sigma_x^2 \sigma_z^3, \sigma_y^1 \sigma_y^2, \sigma_y^1 \sigma_y^2 \sigma_z^3)]$ . The capacity of dense coding channel [64, 65] is  $\chi(\rho^{AB}) = \log_2 D_A + S(\rho^B) - S(\rho^{AB})$  where  $D_A$  is the dimension of Alice's system,  $\rho^B = \text{Tr}_A(\rho^{AB})$  is reduced density matrix of Bob's system with respect to Alice's system and  $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$  is the von Neumann entropy. For the entangled channel  $|\varphi\rangle_{12345}^{(i)}$ ,  $D_A = 2^3$ ,  $S(\rho^B) = 2$ , and  $S(\rho^{AB}) = 0$  which shows  $\chi(\rho^{AB}) = 5$  and thus maximizes the channel capacity. It has been shown earlier that by using a maximally entangled five-qubit GHZ state or a generalized five-qubit W state as a quantum channel, Alice can only send 4-bit information.

Quantum dense coding can also be realized involving a controller between Alice and Bob. By doing so, we compare the transfer efficiency of the channel proposed by us to that of others discussed in section 2. We do this by using three different states of maximally entangled five-qubits

$$\begin{aligned}
 (i) \quad |\Psi\rangle_{12345}^{(1)} &= \frac{1}{2} [|00000\rangle + |10101\rangle + |01110\rangle - |11011\rangle]_{12345} \cdot \\
 (ii) \quad |\Phi\rangle_{12345}^{(1)} &= \frac{1}{2} [|00000\rangle + |10101\rangle + |01110\rangle + |11011\rangle]_{12345} \cdot \\
 (iii) \quad |\varphi\rangle_{12345}^{(10)} &= \frac{1}{2\sqrt{2}} [|00000\rangle - |00101\rangle + |11100\rangle + |11001\rangle \\
 &\quad + |01111\rangle - |01010\rangle + |10011\rangle + |10110\rangle]_{12345} \quad (26)
 \end{aligned}$$

given by Eq. (16), Eq. (17) and Eq. (18), respectively.

The quantum channel  $|\Psi\rangle_{12345}$  is shared by three users Alice (1 and 2), Charlie (3) and Bob (4 and 5). Charlie performs a von Neumann measurement on his share of qubit in the new basis  $(|x_1\rangle_3, |x_2\rangle_3)$  such that

$$\begin{aligned}
 |0\rangle_3 &= \cos \theta |x_1\rangle_3 + \sin \theta |x_2\rangle_3 \quad \text{and} \\
 |1\rangle_3 &= \sin \theta |x_1\rangle_3 - \cos \theta |x_2\rangle_3 \quad (27)
 \end{aligned}$$

where  $\theta = \theta$  if  $0 \leq \theta \leq (\pi)/4$ . For  $\theta$  in the range  $(\pi)/4 \leq \theta \leq (\pi)/2$ , replace  $\theta$  in Eq. [27] by  $(90^\circ - \theta)$ . In this basis,  $|\Psi\rangle_{12345}$  is given by

$$\begin{aligned} |\Psi\rangle_{12345} = & \frac{|x_1\rangle_3}{2} [\cos \theta |0000\rangle + \sin \theta |1001\rangle + \sin \theta |0110\rangle - \cos \theta |1111\rangle]_{1245} \\ & + \frac{|x_2\rangle_3}{2} [\sin \theta |0000\rangle - \cos \theta |1001\rangle - \cos \theta |0110\rangle - \sin \theta |1111\rangle]_{1245}. \end{aligned} \quad (28)$$

In the special case of  $\theta = \pi/4$ , the four qubits are in maximally entangled four-qubit state and Alice can send 4 bits of message to Bob by first encoding the message and then sending 2 qubits of hers to him. However, Charlie can do measurement for any  $\theta$ . (A) If Charlie's result is  $|x_1\rangle_3$  and he communicates his measurement result to Alice then the dense coding process is given schematically in Fig. (3). Alice introduces an auxiliary qubit  $|0\rangle_{aux}$  and does a joint unitary operation  $U_{12aux}$  on her qubits 1, 2 and auxiliary qubit (in the computational basis for qubits 1, 2 and 3). The unitary operation is

$$U_{12aux} = \begin{pmatrix} \frac{\sin \theta}{\cos \theta} & \sqrt{1 - \frac{\sin^2 \theta}{\cos^2 \theta}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{1 - \frac{\sin^2 \theta}{\cos^2 \theta}} & \frac{-\sin \theta}{\cos \theta} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sin \theta}{\cos \theta} & \sqrt{1 - \frac{\sin^2 \theta}{\cos^2 \theta}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{1 - \frac{\sin^2 \theta}{\cos^2 \theta}} & \frac{-\sin \theta}{\cos \theta} \end{pmatrix}. \quad (29)$$

If Charlie's measurement outcome is  $|x_2\rangle_3$ , the unitary operation that Alice will use is  $U'_{12aux}$  where

$$U'_{12aux} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sin \theta}{\cos \theta} & \sqrt{1 - \frac{\sin^2 \theta}{\cos^2 \theta}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{1 - \frac{\sin^2 \theta}{\cos^2 \theta}} & \frac{-\sin \theta}{\cos \theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sin \theta}{\cos \theta} & \sqrt{1 - \frac{\sin^2 \theta}{\cos^2 \theta}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{1 - \frac{\sin^2 \theta}{\cos^2 \theta}} & \frac{-\sin \theta}{\cos \theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (30)$$

If Alice makes a measurement on auxiliary qubit and gets the measurement result  $|0\rangle_{aux}$  (with the probability  $2\sin^2 \theta$ ), she knows that the joint state of four qubits is

in a maximally entangled four-qubit state; however, the measurement result  $|1\rangle_{aux}$  (with the probability  $\cos^2\theta$ ), will confirm that the four-qubits are in GHZ state. In the first case, Alice can encode her message using 16 binary operators from the set  $(I^1, \sigma_x^1, \sigma_y^1, \sigma_z^1)$  and  $(I^2, \sigma_x^2, \sigma_y^2, \sigma_z^2)$  corresponding to her two qubits 1 and 2. She sends her two qubits to Bob who decodes the message by doing a joint measurement on four-qubits based on the entangled state Alice has prepared and thus decodes the original message. However, doing a joint measurement on four-qubits to discriminate 16 orthogonal states is experimentally challenging. Hence the following:

(i) Bob does a joint unitary operation on Alice's 1<sup>st</sup> qubit and his 5<sup>th</sup> qubit given as

$$U_{A_1B_5} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \text{ which evolves the four-qubit state(s) into a four-qubit}$$

GHZ state(s), measures the joint state in the four-qubit GHZ basis and recovers the encoded message.

(ii) Bob applies two C-NOT operations on the state of four qubits keeping Alice's qubits 1 and 2 as controls and his qubits 5 and 4 as targets, respectively which leads to a state of direct products of four qubits; Alice's two qubits are in an entangled state and Bob's two qubits are in a computational basis state. By doing these operations Bob ensures that he differentiates between the subsets of Alice's operations, i.e. if he measures his two qubits in computational basis states and finds  $|00\rangle_{45}$ , he knows that the operation Alice has used to encode the message belongs to the subset  $(I^1 I^1, \sigma_z^1, \sigma_z^2, \sigma_z^1 \otimes \sigma_z^2)$ . Similarly other measurements  $|01\rangle_{45}$ ,  $|10\rangle_{45}$  and  $|11\rangle_{45}$  belong to the subsets  $(\sigma_x^1, \sigma_y^1, \sigma_x^1 \otimes \sigma_z^2, \sigma_y^1 \otimes \sigma_z^2)$ ,  $(\sigma_x^2, \sigma_y^2, \sigma_x^2 \otimes \sigma_z^1, \sigma_y^2 \otimes \sigma_z^1)$  and  $(\sigma_x^1 \otimes \sigma_x^2, \sigma_x^1 \otimes \sigma_y^2, \sigma_y^1, \sigma_x^2, \sigma_y^1 \otimes \sigma_y^2)$ , respectively. Bob uses Bell basis to measure Alice's qubits (which are just a local transformation away from Bell states) and decodes the 4-bit classical message with relative ease.

In the other case, where Alice's measurement result yields  $|1\rangle_{aux}$ , she can still send 3-bit classical message to Bob. Therefore on the average

$$I_{C-A}^{(1)} = 2 \sin^2 \theta + 3 \quad (31)$$

-bit classical message is transferred where the suffix C-A denotes that Charlie informs his measurement result to Alice only.

(B) If Charlie sends his measurement result to Bob and not to Alice, then the process for sending the information is pictorially represented in Fig. (4). Alice does local operations on her qubits and sends them to Bob who, cannot however, do a joint measurement to discriminate all the state of four-qubits as they may or may not be orthogonal. We take Alice's first operational subset  $(I^1 I^1, \sigma_z^1, \sigma_z^2, \sigma_z^1 \otimes \sigma_z^2)$ , as an example, such that the four-qubit state immediately after Alice's operation is  $|\psi\rangle_{1245} = \frac{1}{\sqrt{2}} [\cos \theta |0000\rangle \pm \sin \theta |1001\rangle \pm \sin \theta |0110\rangle \mp \cos \theta |1111\rangle]_{1245}$  and where -ve signs are always in odd numbers. After receiving her qubits Bob does two C-NOT operations as described in the case (A). Alice's qubits are partially entangled, whereas Bob's qubits are in computational basis state such that  $|\psi\rangle_{1245} = \frac{1}{\sqrt{2}} [\cos \theta |00\rangle \pm \sin \theta |01\rangle \pm \sin \theta |10\rangle \mp \cos \theta |11\rangle]_{12} \otimes |00\rangle_{45}$ . By measuring his qubits in the computational basis,



Bob gets Alice's operational subset on her two qubits, thus he needs to differentiate between four states of two-qubit system related to each operational subset. For this Bob introduces an auxiliary qubit in state  $|0\rangle_{aux}$  and performs a joint unitary operation given by Eq. (29) on three qubits (1, 2 and aux) in the computational basis of  $12aux$ , such that

$$\begin{aligned} |\psi'\rangle_{12aux} &= (U_{12aux}) |\psi\rangle_{12aux} \\ &= \sqrt{2} \sin \theta \cdot \frac{1}{2} [|00\rangle \pm |01\rangle \pm |10\rangle \mp |11\rangle]_{12} \otimes |0\rangle_{aux} \\ &\quad + \cos \theta \cdot \sqrt{1 - \frac{\sin^2 \theta}{\cos^2 \theta}} \cdot \frac{1}{\sqrt{2}} [|00\rangle \mp |11\rangle]_{12} \otimes |1\rangle_{aux}. \end{aligned} \quad (32)$$

The average information transfer from Alice to Bob will then be

$$I_{C-B}^{(2)} = 2 \sin^2 \theta + 3 \quad (33)$$

where the suffix C-B denotes that Charlie informs his measurement result to Bob only. It is clear that in both the cases [(A) and (B)] the amount of information transfer, on average, between Alice and Bob is the same. However, in the case of a five-qubit GHZ state the amount of information transferred in controlled manner is  $2 \sin^2 \theta + 2$ .

A similar calculation for information transfer between Alice and Bob using the quantum channel proposed,  $|\varphi\rangle_{12345}^{(i)}$ , shows that irrespective of the measurement basis used by Charlie, Alice is always able to send 4-bit information to Bob using her 2-qubits. Even if Charlie does not inform Alice about the measurement basis used, she is able to send maximum information to Bob using her two qubits such that the information transfer between Alice and Bob is independent of the value of analyzer angle  $\theta$ . At the same time, the information transfer between Alice and Bob using  $|\Phi\rangle_{12345}^{(1)}$  is the same as one obtains in the case of  $|\Psi\rangle_{12345}^{(1)}$ . It is seen that all the five-qubit states used for controlled dense coding possess maximum correlation between the particles, however, the amount of information transfer using different quantum channels is not the same. The entangled set proposed here is advantageous in terms of average information transfer between the sender and the receiver as compared to other five-qubit entangled sets. Thus the representation of a quantum channel used in a communication network and distribution of qubits between different users is an important factor in information processing. A graphical comparison is made in Fig. (5) to compare the efficiency of genuinely entangled five-qubit states discussed here and the five-qubit GHZ states in terms of average information transferred during the process and the analyzer angle  $\theta$ .

For generalized dense coding the  $(2N+1)$ -qubit quantum channel is shared between Alice and Bob such that  $(N+1)$  qubits (1 to  $N+1$ ) are with Alice and rest of the qubits are with Bob. By locally manipulating her qubits, Alice encodes her message and sends her qubits to Bob who, in turn, does the required measurements involving all the qubits and decodes the message. In principle Alice can prepare  $2^{2N+1}$  orthogonal basis states and hence  $2^{2N+1}$  distinguishable messages for Bob. Thus by using the generalized entangled channels Alice can send  $(2N+1)$ -bit information to Bob. Alternatively, the  $(2N+1)$ -qubit state is shared between the three users in communication protocol such

that  $N$  qubits (1 to  $N$ ) are with Alice, one qubit is with Charlie  $[(N + 1)\text{-th}]$  and the rest  $N$  qubits are with Bob ( $N + 2 \rightarrow 2N + 1$ ). Charlie measures his qubit in the basis given by Eq. (27) and either sends his measurement results to Alice or to Bob. There are two instances:

(A) If Charlie sends his measurement results to Alice, she introduces an auxiliary qubit and does a combined unitary transformation on her  $N$  qubits and the auxiliary qubit. After this she measures the state of auxiliary qubit, encodes her message and sends her qubits to Bob who does a joint measurement on  $2N$  qubits and decodes the message. In practice it is really difficult to discriminate multi-qubit states experimentally. As an alternate, Bob can do  $N$  C-NOT operations keeping Alice's qubits 1, 2, 3, ...,  $N$  as controls and his qubits  $(2N + 1), 2N, (2N - 1), \dots, (N + 2)$ , respectively as targets and measure last  $(2N - 2)$  qubits in a computational basis and first two qubits in Bell basis (which are generally a local transformation away from Bell states) and recover the message.

(B) If Charlie sends his measurement results to Bob, Alice encodes her message and sends her qubits to Bob who applies  $N$  C-NOT operations as discussed above, measures last  $(2N - 2)$  qubits in a computational basis and then introduces an auxiliary qubit so that he can discriminate Alice's operation by doing joint unitary transformation on three qubits. After the transformation, Bob measures the state of the auxiliary qubit and discriminates Alice's operations to decode the message.

Although the average information transferred in both the cases is  $[2 \sin^2 \theta + (2N - 1)]$ , the case where Charlie sends his measurement results to Bob is more appealing as the joint transformation that Bob needs to do involves only 3 qubits, however, the one which Alice does involves  $(N + 1)$  qubits. In the case of  $(2N + 1)$ -qubit generalization of  $|\varphi\rangle_{12345}^{(i)}$ , Alice is able to send a  $2N$ -bit information to Bob independent of Charlie's measurement basis.

#### 4. CONCLUSIONS

We have given a criterion to assess the degree of entanglement between qubits/spin-1/2 particles using statistical correlation coefficients as a measure of entanglement. Ursell-Mayer type correlation functions have been suggested to calculate the correlations between multiple particles which are extensions to the two-particle functions. The use of Ursell-Mayer type correlation functions to calculate the entanglement between multi-particles is an attempt to generalize the definition of degree of entanglement in multi-particle systems irrespective of the number of particles involved. The criterion is shown to be unique in characterizing different entangled systems in different families. It has been shown that the local transformations between two-states can be established by visual examination of the non-zero correlation coefficients associated with the systems under study. The criterion developed here is compared with the existing entanglement norms for two and three particle maximally and non-maximally entangled systems and found to be in excellent match.

We have proposed a maximally and genuinely entangled five-particle quantum channel and described an efficient theoretical approach for direct quantum teleportation of multi-particle information. The process discussed here overcomes the difficulty of getting null results in half of the projections when dealing with odd number of particles comprising a quantum channel. Teleportation using a controller has also been shown to be effective using appropriate projection basis. Physical realization of states and quantum teleportation protocols are analyzed using standard quantum gates and circuit diagrams. Quantum dense coding using the state proposed has been shown to be optimal. For controlled dense coding process it has been observed that unlike cluster and  $|\chi\rangle$  type of states where amount of information transfer depends on the analyzer angle used by the controller, the information transfer using the state proposed is always maximum irrespective of the measurement basis used by the controller. A comparison between average information transferred in the case of the state proposed here and other five-qubit states including GHZ state has been made through a plot at various analyzer angles to have more insight into the entanglement properties and representation of quantum channel used. It has been observed that for higher number of qubits when a controller is involved, it is desirable that measurement results be transmitted to the receiver and not to the sender.

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## Appendix A: Correlation coefficients for five-particles

The correlation coefficients  $C_{\alpha\beta\gamma\delta\kappa}^{12345}$  for five-particle systems are given by

$$\begin{aligned}
C_{\alpha\beta\gamma\delta\kappa}^{12345} = & \langle \sigma_\alpha^1 \sigma_\beta^2 \sigma_\gamma^3 \sigma_\delta^4 \sigma_\kappa^5 \rangle - \langle \sigma_\alpha^1 \rangle [C_{\beta\gamma\delta\kappa}^{2345}] - \langle \sigma_\beta^2 \rangle [C_{\alpha\gamma\delta\kappa}^{1345}] - \langle \sigma_\gamma^3 \rangle [C_{\alpha\beta\delta\kappa}^{1245}] \\
& - \langle \sigma_\delta^4 \rangle [C_{\alpha\beta\gamma\kappa}^{1235}] - \langle \sigma_\kappa^5 \rangle [C_{\alpha\beta\gamma\delta}^{1234}] - \langle \sigma_\alpha^1 \sigma_\beta^2 \rangle \langle \sigma_\gamma^3 \sigma_\delta^4 \sigma_\kappa^5 \rangle \\
& - \langle \sigma_\alpha^1 \sigma_\gamma^3 \rangle \langle \sigma_\beta^2 \sigma_\delta^4 \sigma_\kappa^5 \rangle - \langle \sigma_\alpha^1 \sigma_\delta^4 \rangle \langle \sigma_\beta^2 \sigma_\gamma^3 \sigma_\kappa^5 \rangle - \langle \sigma_\alpha^1 \sigma_\kappa^5 \rangle \langle \sigma_\beta^2 \sigma_\gamma^3 \sigma_\delta^4 \rangle \\
& - \langle \sigma_\beta^2 \sigma_\gamma^3 \rangle \langle \sigma_\alpha^1 \sigma_\delta^4 \sigma_\kappa^5 \rangle - \langle \sigma_\beta^2 \sigma_\delta^4 \rangle \langle \sigma_\alpha^1 \sigma_\gamma^3 \sigma_\kappa^5 \rangle - \langle \sigma_\beta^2 \sigma_\kappa^5 \rangle \langle \sigma_\alpha^1 \sigma_\gamma^3 \sigma_\delta^4 \rangle \\
& - \langle \sigma_\gamma^3 \sigma_\delta^4 \rangle \langle \sigma_\alpha^1 \sigma_\beta^2 \sigma_\kappa^5 \rangle - \langle \sigma_\gamma^3 \sigma_\kappa^5 \rangle \langle \sigma_\alpha^1 \sigma_\beta^2 \sigma_\delta^4 \rangle - \langle \sigma_\delta^4 \sigma_\kappa^5 \rangle \langle \sigma_\alpha^1 \sigma_\beta^2 \sigma_\gamma^3 \rangle \\
& + \langle \sigma_\alpha^1 \rangle \langle \sigma_\beta^2 \sigma_\gamma^3 \rangle \langle \sigma_\delta^4 \sigma_\kappa^5 \rangle + \langle \sigma_\alpha^1 \rangle \langle \sigma_\beta^2 \sigma_\delta^4 \rangle \langle \sigma_\gamma^3 \sigma_\kappa^5 \rangle + \langle \sigma_\alpha^1 \rangle \langle \sigma_\beta^2 \sigma_\kappa^5 \rangle \langle \sigma_\gamma^3 \sigma_\delta^4 \rangle \\
& + \langle \sigma_\beta^2 \rangle \langle \sigma_\alpha^1 \sigma_\gamma^3 \rangle \langle \sigma_\delta^4 \sigma_\kappa^5 \rangle + \langle \sigma_\beta^2 \rangle \langle \sigma_\alpha^1 \sigma_\delta^4 \rangle \langle \sigma_\gamma^3 \sigma_\kappa^5 \rangle + \langle \sigma_\beta^2 \rangle \langle \sigma_\alpha^1 \sigma_\kappa^5 \rangle \langle \sigma_\gamma^3 \sigma_\delta^4 \rangle \\
& + \langle \sigma_\gamma^3 \rangle \langle \sigma_\alpha^1 \sigma_\beta^2 \rangle \langle \sigma_\delta^4 \sigma_\kappa^5 \rangle + \langle \sigma_\gamma^3 \rangle \langle \sigma_\alpha^1 \sigma_\delta^4 \rangle \langle \sigma_\beta^2 \sigma_\kappa^5 \rangle + \langle \sigma_\gamma^3 \rangle \langle \sigma_\alpha^1 \sigma_\kappa^5 \rangle \langle \sigma_\beta^2 \sigma_\delta^4 \rangle \\
& + \langle \sigma_\delta^4 \rangle \langle \sigma_\alpha^1 \sigma_\beta^2 \rangle \langle \sigma_\gamma^3 \sigma_\kappa^5 \rangle + \langle \sigma_\delta^4 \rangle \langle \sigma_\alpha^1 \sigma_\gamma^3 \rangle \langle \sigma_\beta^2 \sigma_\kappa^5 \rangle + \langle \sigma_\delta^4 \rangle \langle \sigma_\alpha^1 \sigma_\kappa^5 \rangle \langle \sigma_\beta^2 \sigma_\gamma^3 \rangle \\
& + \langle \sigma_\kappa^5 \rangle \langle \sigma_\alpha^1 \sigma_\beta^2 \rangle \langle \sigma_\gamma^3 \sigma_\delta^4 \rangle + \langle \sigma_\kappa^5 \rangle \langle \sigma_\alpha^1 \sigma_\gamma^3 \rangle \langle \sigma_\beta^2 \sigma_\delta^4 \rangle + \langle \sigma_\kappa^5 \rangle \langle \sigma_\alpha^1 \sigma_\delta^4 \rangle \langle \sigma_\beta^2 \sigma_\gamma^3 \rangle
\end{aligned}$$

$$-6 \langle \sigma_{\alpha}^1 \rangle \langle \sigma_{\beta}^2 \rangle \langle \sigma_{\gamma}^3 \rangle \langle \sigma_{\delta}^4 \rangle \langle \sigma_{\kappa}^5 \rangle.$$

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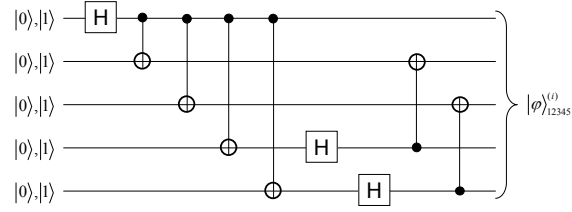
- (i) Quantum circuit to prepare the set of states  $|\varphi\rangle_{12345}^{(i)}$  represented by Eq. (18).
- (ii) Quantum network to teleport an arbitrary two qubit state  $|\phi\rangle_{12} = [|00\rangle_{12} + |01\rangle_{12} + |10\rangle_{12} + |11\rangle_{12}]$  using  $|\varphi\rangle_{34567}^{(10)}$  as quantum channel.
- (iii) Controlled dense coding of five-qubit state  $|\Psi\rangle_{12345}^{(1)}$  with controller-sender interface.
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- (v) Comparison of the efficiency of information transfer between states given by Eq. (26) and the five qubit GHZ state.

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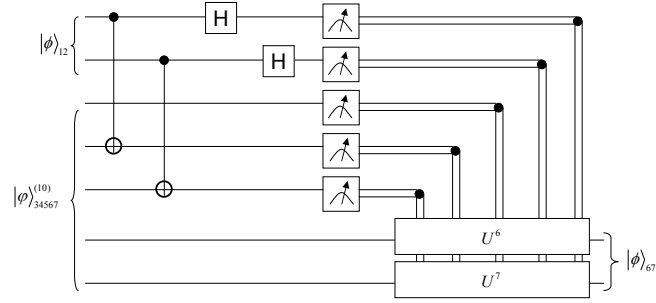
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Figure 1



**Figure 1.** Quantum circuit to prepare the set of states  $|\varphi\rangle_{12345}^{(i)}$ .

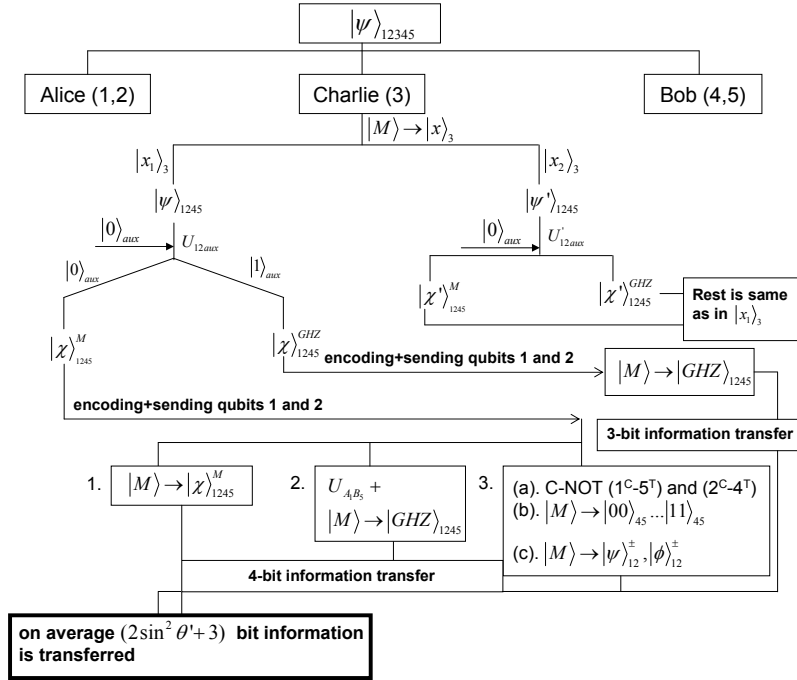
Figure 2



**Figure 2.** Quantum network to teleport an arbitrary two particle state through  $|\varphi\rangle_{34567}^{(1)}$ .

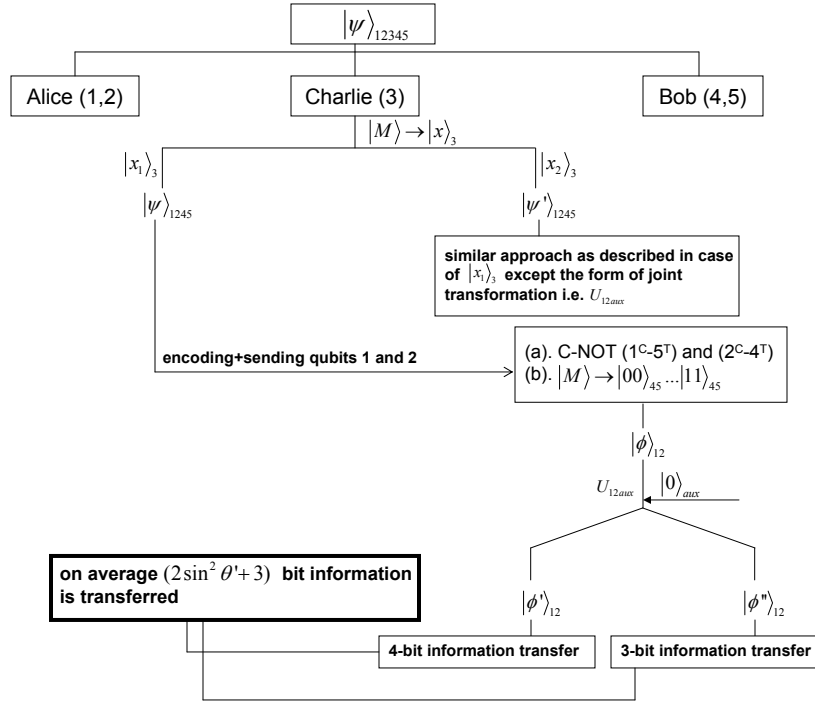


Figure 3



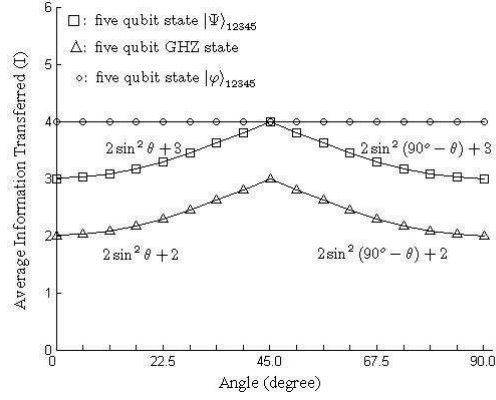
**Figure 3.** Schematic representation of controlled dense coding process with controller-sender interface.

Figure 4



**Figure 4.** Schematic representation of controlled dense coding process with controller-receiver interface.

Figure 5



**Figure 5.** Comparison of the efficiency of information transfer between states given by Eq. (26) and the five qubit GHZ state.